



# Multi-objective linear programming with interval coefficients

## A fuzzy set based approach

Seyed Hossein Razavi Hajiagha

*Department of Systemic and Productivity Studies,  
Institute for Trade Studies and Research, Tehran, Iran, and*

Hannan Amoozad Mahdiraji and Shide Sadat Hashemi  
*Department of Management, Kashan Branch, Islamic Azad University,  
Kashan, Iran*

### Abstract

**Purpose** – The purpose of this paper is to extend a methodology for solving multi-objective linear programming (MOLP) problems, when the objective functions and constraints coefficients are stated as interval numbers.

**Design/methodology/approach** – The approach proposed in this paper for the considered problem is based on the maximization of the sum of membership degrees which are defined for each objective of multi objective problem. These membership degrees are constructed based on the deviation from optimal solutions of individual objectives. Then, the final model based on membership degrees is itself an interval linear programming which can be solved by current methods.

**Findings** – The efficiency of the solutions obtained by the proposed method is proved. It is shown that the obtained solution by the proposed method for an interval multi objective problem is Pareto optimal.

**Research limitations/implications** – The proposed method can be used in modeling and analyzing of uncertain systems which are modeled in the context of multi objective problems and in which required information is ill defined.

**Originality/value** – The paper proposed a novel and well-defined algorithm to solve the considered problem.

**Keywords** Multi objective linear programming, Interval numbers, Order relations, Membership degree, Pareto optimality, Linear programming, Fuzzy logic, Programming and algorithm theory, Modelling, Decision making

**Paper type** Research paper



### 1. Introduction

The main purpose of decision making methods is to provide a holistic criterion to evaluate the utility of an action or choice based on multiple criteria. Practical and real world decision making problems often must satisfy several goals which are sometimes conflicting and inconsistent. Such a problem is the subject of multiple criteria decision making (MCDM) methods. This class is further divided into multi-objective decision making (MODM) and multi-attribute decision making (MADM) (Climaco, 1997). MODM problems are optimization type problems. Optimization is an inherent behavior of all human, physical, and natural systems. The major cause of this behavior

is limitation in resources (Nocedal and Wright, 1999). MODM is a generalization of more traditional, single-objective approach. Cohon (2004) believed that considering more than one objective have the following advantages:

- Promotes more appropriate roles for the participants in the decision-making process.
- A wider range of alternatives is usually identified.
- Models or the analyst's perception of a problem will be more realistic.

These aspects provide a widespread range of methods and algorithms to deal with and to solve MODM problems. The main focus of this paper is on a class of multi-objective linear programming (MOLP) problems which their parameters and variables information are specified uncertainly. Classical MOLP solving procedures are extended based on the assumption of crispness. In fact, they suppose that all of the data that are needed for modeling and solving an MOLP are specified. But in practice, this assumption is often violated. Uncertainty can occur because:

- the information is partial; or
- the information is approximate (Yovits, 1984).

Different frameworks are introduced in response for modeling and analyzing the uncertainty of systems. Liu and Lin (2006) classified the uncertainty frameworks into three distinct fields:

- (1) probability and statistics;
- (2) fuzzy sets theory; and
- (3) grey systems theory.

A key note in modeling uncertain systems is that a complicated model is not always necessary to deal with incomplete information and inaccurate data (Liu *et al.*, 2012). Interval numbers are a framework which aids to avoid such an unnecessary complexity. Moore (1966) originally introduced interval numbers. An interval number is a number whose exact value is unknown, but a range within which the value lies is known (Moore *et al.*, 2009). An interval number allows analyst to make his/her approximations about parameters on an interval rather than a crisp number. This flexibility caused interval numbers to had great applications in optimization problems.

The literature on the applications of interval number in decision making problems is so wide. The focus of this paper, however, is just on the application of interval number in mathematical optimization problems. In the field of interval linear programming, Ishibuchi and Tanaka (1990), Inuiguchi and Sakawa (1995), Chanas and Kuchta (1996), Chinneck and Ramadan (2000), Sengupta *et al.* (2001) and Chen *et al.* (2004) developed different procedures to deal with these problems. In real world applications, problems often deal with multiple conflict objectives which may be modeled in the context of multi-objective problems. For a more detailed discussion of multi-objective programming see Tamiz (1996), Cohon (2004) and Barichard *et al.* (2009). Like any other decision making problem, multi-objective models deal with uncertainty in their information. In this context, some frameworks are proposed to solve multi-objective problems with interval parameters. Das *et al.* (1999), Ida (1999, 2000a, b, 2005) and Wang and Wang (2001a, b) are some works on interval MOLP. Oliveira and Antunes (2007) presented an overview on

some current procedures of interval MOLP. The above methods have some disadvantages, like a considerable computational efforts (Ida, 1999), sensitivity to the ordering of rows in ray generation methods (Ida, 2000a, b, 2005), and computational inefficiency when the number of objective functions increases (Wang and Wang, 2001a, b), which encourages to develop a new framework for IMOLP problems.

In this paper, a new framework is developed to treat with interval MOLP problems. In the first step, the proposed method applied a compromise programming based logic to obtain the best and worst solutions of each interval objectives. In the second phase, the proposed method tries to find the better compromise solution which simultaneously satisfied different objectives based on a fuzzy set approach. The rest of the paper is organized as follows. A brief overview of grey theory is presented in Section 2. A definition of interval MOLP problem is given in Section 3. The proposed approach and its related issues are explained in Section 4. In Section 5, a numerical example is solved by the proposed procedure. Finally, Section 6 consists of conclusions.

## 2. Interval numbers

An interval number is a number whose exact value is unknown, but a range within which the value lies is known (Moore *et al.*, 2009). Interval number is a number with both lower and upper bounds,  $\tilde{x} \in [\underline{x}, \bar{x}]$ , where  $\underline{x} \leq \bar{x}$ . The main arithmetic operations can be defined on interval numbers. Let  $\tilde{x}_1 = [\underline{x}_1, \bar{x}_1]$  and  $\tilde{x}_2 = [\underline{x}_2, \bar{x}_2]$  be two interval numbers. The following operations can be defined (Moore *et al.*, 2009):

$$\tilde{x}_1 + \tilde{x}_2 = [\underline{x}_1 + \underline{x}_2, \bar{x}_1 + \bar{x}_2] \quad (1)$$

$$\tilde{x}_1 - \tilde{x}_2 = [\underline{x}_1 - \bar{x}_2, \bar{x}_1 - \underline{x}_2] \quad (2)$$

$$\tilde{x}_1 \times \tilde{x}_2 = \left[ \min(\underline{x}_1 \underline{x}_2, \underline{x}_1 \bar{x}_2, \bar{x}_1 \underline{x}_2, \bar{x}_1 \bar{x}_2), \max(\underline{x}_1 \underline{x}_2, \underline{x}_1 \bar{x}_2, \bar{x}_1 \underline{x}_2, \bar{x}_1 \bar{x}_2) \right] \quad (3)$$

$$\tilde{x}_1 \div \tilde{x}_2 = [\underline{x}_1, \bar{x}_1] \times \left[ \frac{1}{\bar{x}_2}, \frac{1}{\underline{x}_2} \right] \quad (4)$$

When  $\tilde{x} \in [\underline{x}, \bar{x}]$  is an interval number, its absolute value is the maximum of the absolute value of its endpoints:  $|\tilde{x}| = \max(|\underline{x}|, |\bar{x}|)$  (Huang, 1994).

The center,  $x_C$ , and width,  $x_W$  of a grey number  $\tilde{x} \in [\underline{x}, \bar{x}]$  is defined as follows:

$$x_C = \frac{1}{2}(\underline{x} + \bar{x}) \quad (5)$$

$$x_W = \frac{1}{2}(\bar{x} - \underline{x}) \quad (6)$$

It is easily verifiable that  $\bar{x} = x_C + x_W$  and  $\underline{x} = x_C - x_W$ . Ishibuchi and Tanaka (1990) defined the following order relations between intervals.

**Definition 1.** If  $\tilde{x} = [\underline{x}, \bar{x}]$  and  $\tilde{y} = [\underline{y}, \bar{y}]$  are two interval numbers, then the order relation  $\leq_{LR}$  is defined as:

$$\tilde{x} \leq_{LR} \tilde{y} \text{ iff } \underline{x} \leq \underline{y} \text{ and } \bar{x} \leq \bar{y} \quad (7)$$

$$\tilde{x} <_{LR} \tilde{y} \text{ iff } \tilde{x} \leq_{LR} \tilde{y} \text{ and } x \neq y \quad (8)$$

*Definition 2.* The order relation  $\leq_{CW}$  between two grey numbers  $\tilde{x}$  and  $\tilde{y}$  is defined as:

$$\tilde{x} \leq_{CW} \tilde{y} \text{ iff } x_C \leq y_C \text{ and } x_W \geq y_W \quad (9)$$

$$\tilde{x} <_{CW} \tilde{y} \text{ iff } \tilde{x} \leq_{CW} \tilde{y} \text{ and } x \neq y \quad (10)$$

The order relations  $\leq_{CW}$  and  $\leq_{LR}$  never conflicts with each other. Similarly, Ishibuchi and Tanaka (1990) defined  $\leq_{LR}^*$  and  $\leq_{CW}^*$  as follows.

*Definition 3.* If  $\tilde{x} = [\underline{x}, \bar{x}]$  and  $\tilde{y} = [\underline{y}, \bar{y}]$  are two interval numbers, then the order relation  $\leq_{LR}^*$  is defined as:

$$\tilde{x} \leq_{LR}^* \tilde{y} \text{ iff } \underline{x} \leq \underline{y} \text{ and } \bar{x} \leq \bar{y} \quad (11)$$

$$\tilde{x} <_{LR}^* \tilde{y} \text{ iff } \tilde{x} \leq_{LR}^* \tilde{y} \text{ and } x \neq y \quad (12)$$

*Definition 4.* The order relation  $\leq_{CW}^*$  between two grey numbers  $\tilde{x}$  and  $\tilde{y}$  is defined as:

$$\tilde{x} \leq_{CW}^* \tilde{y} \text{ iff } x_C \leq y_C \text{ and } x_W \leq y_W \quad (13)$$

$$\tilde{x} <_{CW}^* \tilde{y} \text{ iff } \tilde{x} \leq_{CW}^* \tilde{y} \text{ and } x \neq y \quad (14)$$

### 3. Interval multi-objective linear programming

An interval MOLP problem can be stated as follows:

$$\begin{aligned} \max(\min) \tilde{Z}_1 &= \sum_{j=1}^n \tilde{c}_{1j} \tilde{x}_j \\ \max(\min) \tilde{Z}_2 &= \sum_{j=1}^n \tilde{c}_{2j} \tilde{x}_j \\ &\vdots \\ \max(\min) Z_k &= \sum_{j=1}^n \tilde{c}_{kj} \tilde{x}_j \end{aligned} \quad (15)$$

*S.T.*

$$\sum_{j=1}^n \tilde{a}_{ij} \tilde{x}_j \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, \quad i = 1, 2, \dots, m$$

$$\tilde{x}_j \geq 0, \quad j = 1, 2, \dots, n$$

where:

$\tilde{x}_j \in [\underline{x}_j, \bar{x}_j]$ ,  $j = 1, 2, \dots, n$  are the interval decision variables.

$\tilde{c}_{ij} \in [\underline{c}_{ij}, \bar{c}_{ij}]$ ,  $i = 1, 2, \dots, k$ ;  $j = 1, 2, \dots, n$  are interval objective functions' coefficients.

$\tilde{a}_{ij} \in [\underline{a}_{ij}, \bar{a}_{ij}]$ ,  $i = 1, 2, \dots, m$ ;  $j = 1, 2, \dots, n$  and  $\tilde{b}_i \in [\underline{b}_i, \bar{b}_i]$ ,  $i = 1, 2, \dots, m$  are interval constraints coefficients and right hand sides.

The problem here is to find the optimal solution for above interval MOLP problem.

#### 4. Fuzzy approach to solve interval MOLP

The proposed method to solve the interval MOLP (15) is a multi-stage procedure. In the first step, the problem (15) is decomposed to a set of  $k$  interval linear programming problems. Each problem optimizes one of the  $k$  objective functions regard to constraints set. For a given objective  $l$ , this problem is as follows:

$$\begin{aligned} \max(\min) \tilde{Z}_l &= \sum_{j=1}^n \tilde{c}_{lj} \tilde{x}_j, \quad l = 1, 2, \dots, k \\ \text{S.T.} \\ \sum_{j=1}^n \tilde{a}_{ij} \tilde{x}_j &\begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, \quad i = 1, 2, \dots, m \\ \tilde{x}_j &\geq 0, \quad j = 1, 2, \dots, n \end{aligned} \quad (16)$$

The problem (16) is an interval linear programming model which can be solved by current methods. Razavi Hajiagha *et al.* (2012) proposed a method to solve such problems. This method transformed an interval linear programming method into two equivalent models for its lower bound and upper bound. Suppose that  $K^+$  includes those variables that their objective coefficients have both positive lower and upper bound,  $K^-$  includes those variables that their objective coefficients have both negative lower and upper bound and  $K^0$  includes those variables that their objective coefficients have different sign and contain zero in their intervals. Then, the objective function  $\tilde{Z}_l$  can be written as follows:

$$\tilde{Z}_l(x) = \left[ \begin{aligned} &\sum_{j \in K^+} \underline{c}_{lj}^+ \underline{x}_j^+ + \sum_{j \in K^-} \underline{c}_{lj}^- \underline{x}_j^- + \sum_{j \in K^0} \underline{c}_{lj}^0 \underline{x}_j^0, \\ &\sum_{j \in K^+} \bar{c}_{lj}^+ \bar{x}_j^+ + \sum_{j \in K^-} \bar{c}_{lj}^- \bar{x}_j^- + \sum_{j \in K^0} \bar{c}_{lj}^0 \bar{x}_j^0 \end{aligned} \right] \quad (17)$$

Also, the left hand side (LHS) of each constraint can be written as  $\sum_{j=1}^n \tilde{a}_{ij} \cdot \tilde{x}_j$ , where  $\tilde{a}_{ij} \in [\underline{a}_{ij}, \bar{a}_{ij}]$ . The extended form of LHS is as follows:

$$\sum_{j \in K^+} \tilde{a}_{ij}^+ \cdot \tilde{x}_j^+ + \sum_{j \in K^-} \tilde{a}_{ij}^- \cdot \tilde{x}_j^- + \sum_{j \in K^0} \tilde{a}_{ij}^0 \cdot \tilde{x}_j^0 \quad (18)$$

where,  $\tilde{a}_{ij}^+$ ,  $\tilde{a}_{ij}^-$ , and  $\tilde{a}_{ij}^0$  are the associated coefficients of the variables in  $K^+$ ,  $K^-$ , and  $K^0$  sets. The equation (18) is an interval number which can be shown as  $[\underline{LHS}_i, \bar{LHS}_i]$ ,

using the arithmetic operations of interval numbers. Now, the  $i$ th constraint can be justified as follows:

- If  $[\underline{LHS}_i, \overline{LHS}_i] \leq [\underline{b}_i, \bar{b}_i]$ , then according to the order relation  $\leq_{RC}^*$ , it is transformed into:

$$\overline{LHS}_i \leq \bar{b}_i \quad (19)$$

$$(LHS_i)_C \leq b_C \quad (20)$$

- If  $[\underline{LHS}_i, \overline{LHS}_i] \geq [\underline{b}_i, \bar{b}_i]$ , then according to the order relation  $\leq_{LC}$ , it is transformed into:

$$\underline{LHS}_i \geq \underline{b}_i \quad (21)$$

$$(LHS_i)_C \geq b_C \quad (22)$$

- If  $[\underline{LHS}_i, \overline{LHS}_i] = [\underline{b}_i, \bar{b}_i]$ , then it is transformed into:

$$\underline{LHS}_i = \underline{b}_i \quad (23)$$

$$\overline{LHS}_i = \bar{b}_i \quad (24)$$

It can be shown that there is a direct relationship between the above relations with the notion of possibility degree, introduced by Li *et al.* (2007) (Razavi Hajiagha *et al.*, 2012).

If the original objective is to minimize  $\tilde{Z}_l, l \in \{1, 2, \dots, k\}$ , the solution of model (16) can be determined as the set of Pareto optimal solutions of the following bi-objective problem:

$$\text{Min}(z_{lC}(x), \bar{z}_l(x)) \quad (25)$$

where,  $\bar{z}_l(x)$  is the upper bound of interval function  $\tilde{z}_l(x)$  and  $z_{lC}(x)$  is its center. If the original objective is to maximize  $\tilde{Z}_l, l \in \{1, 2, \dots, k\}$ , the solution of model (16) can be determined as the set of Pareto optimal solutions of the following bi-objective problem:

$$\text{Max}(\underline{z}_l(x), z_{lC}(x)) \quad (26)$$

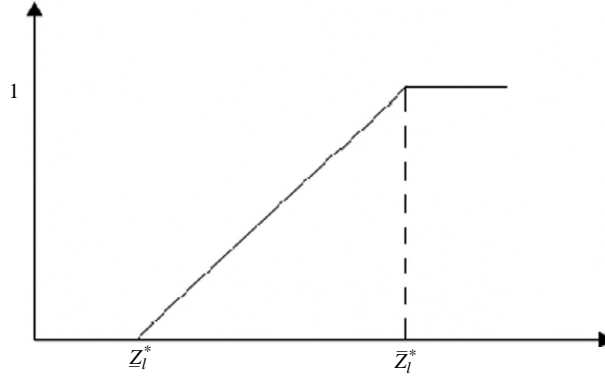
In equation (26) also  $\underline{z}_l(x)$  is the lower bound of interval function  $\tilde{z}_l(x)$ . Finally, by solving the equation (16) based on equations (17)-(26) for each objective function, the range of optimal objective functions are determined as  $\tilde{z}_l^* \in [\underline{z}_l^*, \bar{z}_l^*], l = 1, 2, \dots, k$ .

Now, consider the  $l$ th objective again. If this objective is a maximization type, its membership function can be defined as follows:

$$\tilde{\mu}_l(x) = \begin{cases} 1 & \text{if } \tilde{z}_l(x) \geq \bar{z}_l^* \\ \frac{\tilde{z}_l(x) - \underline{z}_l^*}{\bar{z}_l^* - \underline{z}_l^*} & \text{if } \tilde{z}_l(x) \leq \bar{z}_l^* \end{cases} \quad (27)$$

where the increasing of  $\tilde{z}_l(x)$  will increase the membership degree  $\tilde{\mu}_l(x)$ . This membership function is shown in Figure 1.

**Figure 1.**  
Membership function for  
maximization objectives



In a same way, for minimization type objective, the membership function can be defined as follows:

$$\tilde{\mu}_l(x) = \begin{cases} 1 & \text{if } \tilde{z}_l(x) \leq \underline{z}_l^* \\ \frac{\bar{z}_l^* - \tilde{z}_l(x)}{\bar{z}_l^* - \underline{z}_l^*} & \text{if } \underline{z}_l^* \leq \tilde{z}_l(x) \end{cases} \quad (28)$$

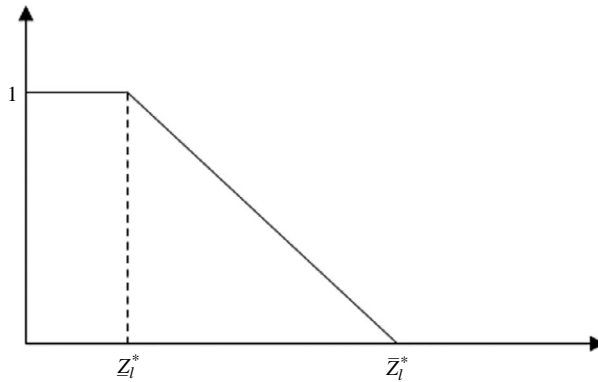
where the decreasing of  $\tilde{z}_l(x)$  will increase the membership degree  $\tilde{\mu}_l(x)$ . Figure 2 shows this membership function.

Lemma 1 shows an important fact in modeling process.

*Lemma 1.* From equation (27) it always holds that  $\tilde{\mu}_l(x) \leq 1$ .

*Proof.* Suppose that  $\tilde{\mu}_l(x) \geq 1$ . Then based on equation (27) it follows that:

$$\tilde{\mu}_l(x) \geq 1 \Rightarrow \frac{\tilde{z}_l(x) - \underline{z}_l^*}{\bar{z}_l^* - \underline{z}_l^*} \geq 1 \Rightarrow \tilde{z}_l(x) - \underline{z}_l^* \geq \bar{z}_l^* - \underline{z}_l^* \Rightarrow \tilde{z}_l(x) \geq \bar{z}_l^*$$



**Figure 2.**  
Membership function for  
minimization objectives

According to equations (21) and (22):

$$\bar{z}_l(x) \geq \bar{z}_l^* : \begin{cases} \underline{z}_l(x) \geq \bar{z}_l^* \\ z_{lc}(x) \geq \bar{z}_l^* \end{cases}$$

From  $\underline{z}_l(x) \geq \bar{z}_l^*$  it required that  $\bar{z}_l(x) \geq \bar{z}_l^*$  which contradicts with the optimality of  $\bar{z}_l^*$ . It completed the proof.

A similar lemma can be proved for equation (28). Then, the final step is to formulate the interval MOLP problem as follows:

$$\begin{aligned} & \max \{ \tilde{\mu}_1(x), \tilde{\mu}_2(x), \dots, \tilde{\mu}_k(x) \} \\ & \tilde{\mu}_l(x) \leq 1, \quad l = 1, 2, \dots, k \\ & \sum_{j=1}^n \tilde{a}_{ij} \tilde{x}_j \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, \\ & i = 1, 2, \dots, m \\ & \tilde{x}_j \geq 0, j = 1, 2, \dots, n \\ & \tilde{\mu}_l(x) : \text{unrestricted}, \quad l = 1, 2, \dots, k \end{aligned} \quad (29)$$

This problem is transformed into the following equivalent form:

$$\begin{aligned} & \max \sum_{l=1}^k \tilde{\mu}_l(x) \\ & \tilde{\mu}_l(x) \leq 1 \\ & \sum_{j=1}^n \tilde{a}_{ij} \tilde{x}_j \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, \quad i = 1, 2, \dots, m \\ & \tilde{x}_j \geq 0, \quad j = 1, 2, \dots, n \\ & \tilde{\mu}_l(x) : \text{unrestricted}, \quad l = 1, 2, \dots, k \end{aligned} \quad (30)$$

It is notable that if decision maker has some preemptive preferences over different objectives, the objective function in model (30) can be replaced by a weighted sum function, i.e.  $\max \sum_{l=1}^k w_l \tilde{\mu}_l(x)$ , where  $w_l$ ,  $l = 1, 2, \dots, k$  are the weights of objective functions such that  $w_l \geq 0$ ,  $l = 1, 2, \dots, k$ .

The model (30) itself is an interval linear programming problem which can be solved by Razavi Hajiagha *et al.* (2012) method. It can be shown that the optimal solution obtained by model (30) is a fuzzy efficient solution for the problem (29) and consequently, a Pareto optimal solution to the interval MOLP problem (15). The notions of Pareto optimal and fuzzy efficient solution are defined as follows.



*Definition 5.* A decision plan  $x^0 \in X$  ( $X$  is the feasible space of interval MOLP problem) is said to be a Pareto optimal solution to the interval MOLP problem (15) if and only if there does not exist another  $y \in X$  such that  $\tilde{z}_l(y) \geq \tilde{z}_l(x^0)$  for all  $l$  and  $\tilde{z}_p(y) > \tilde{z}_p(x^0)$  for at least one  $p$ .  $\square$

*Definition 6.* Suppose that the feasible set of a fuzzy multi-objective linear programming problem is  $X$ . A decision plan  $x^0 \in X$  is said to be a fuzzy-efficient solution to the problem (29) if and only if there does not exist another  $y \in X$  such that  $\tilde{\mu}_l(y) \geq \tilde{\mu}_l(x^0)$  for all  $l$  and  $\tilde{\mu}_p(y) \geq \tilde{\mu}_p(x^0)$  for at least one  $p$  (Werners, 1987a, b; Jimenez and Bilbao-Terol, 2009).

*Lemma 2.* Let  $x^0$  be an optimal solution of problem (30). Then  $x^0$  is a fuzzy efficient solution to the fuzzy MOLP (29).

*Proof.* On the contrary, suppose that  $x^0$  is not a fuzzy efficient solution to problem (29). Therefore, there is another solution  $y$  such that  $\tilde{\mu}_l(y) \geq \tilde{\mu}_l(x^0)$  for all  $l$  and  $\tilde{\mu}_p(y) \geq \tilde{\mu}_p(x^0)$  for at least one  $p$ . Consequently,  $\sum_{l=1}^k \tilde{\mu}_l(y) \geq \sum_{l=1}^k \tilde{\mu}_l(x^0)$  and  $x^0$  is not an optimal solution to the problem (30), a contradiction that complete the proof.  $\square$

*Lemma 3.* Let  $x^0$  be a fuzzy efficient solution to the fuzzy MOLP (29). Then  $x^0$  is a Pareto optimal solution to the interval MOLP (15).

*Proof.* According to the proof of the Lemma 1, the fuzzy efficiency of  $x^0$  to problem (29) means that there does not exist a solution  $y$  such that  $\tilde{\mu}_l(y) \geq \tilde{\mu}_l(x^0)$  for all  $l$  and  $\tilde{\mu}_p(y) \geq \tilde{\mu}_p(x^0)$  for at least one  $p$ . It is enough to show that this condition is equivalent to the definition of Pareto optimality in Definition 5. In fact, it must be shown that  $\tilde{\mu}_l(x^0) \geq \tilde{\mu}_l(y)$  is equivalent to say that  $\tilde{z}_l(x^0) \geq \tilde{z}_l(y)$  (when objective function is maximization) which is obvious from the definition of membership functions in equations (27) and (28). Since there is not a solution that violates the fuzzy efficiency of  $x^0$  to the problem (29), and then there does not exist another solution that violates the Pareto optimality of  $x^0$  to the problem (15) and it complete the proof.

Therefore, solving the problem (30) determines the Pareto optimal solution of an interval MOLP problem.

Oliveira and Antunes (2007) comprehensively reviewed the advantages and disadvantages of other interval MOLP methods, from different points include their required computational efforts. Some methods, like Ida (1999), are based on enumeration concept which is computationally burden. Also, the exponential growth of the number of objective functions in the sub problems of F-cone algorithm (Steuer, 1980, 1986) led it beyond the acceptable computational limits. On the other hand, the Inuiguchi and Kume (1991) proposed a method which solved four formulations based on goal programming, which two formulations are non-convex problems and as they proposed, can be solved by branch and bound algorithm. About the computational efficiency of the proposed method it can be noted that the proposed algorithm for a problem consisting  $k$  objectives, includes solving  $(2k + 1)$  linear programming problems and a bi-objective linear model which can be easily solved by one of the current methods (i.e. the compromise programming (Zeleny, 1976) or goal programming (Charnes and Cooper, 1959)). In fact, the proposed method does not require solving any non-convex or complex problem. Also, the number of linear programs is increased linearly.  $\square$

## 5. Numerical example

In this section, three numerical examples of interval MOLP problems are discussed.

*Example 1.* Consider the following interval MOLP:

$$\begin{aligned}
 \text{Max} Z_1 &= [1, 3]\tilde{x}_1 + [-1, 1.5]\tilde{x}_2 \\
 \text{Max} Z_2 &= [0.5, 2]\tilde{x}_1 + [-1.5, -1]\tilde{x}_2 \\
 \text{S.T.} \\
 [1, 2]\tilde{x}_1 + [1.5, 3]\tilde{x}_2 &\leq [4, 6] \\
 [1, 3]\tilde{x}_1 + [2.5, 3.5]\tilde{x}_2 &\leq 12 \\
 \tilde{x}_1, \tilde{x}_2 &\geq 0
 \end{aligned} \tag{31}$$

The problem (29) is decomposed to two equivalent interval linear programming.

*Problem 1.*

$$\begin{aligned}
 \text{Max} Z_1 &= [1, 3]\tilde{x}_1 + [-1, 1.5]\tilde{x}_2 \\
 \text{S.T.} \\
 [1, 2]\tilde{x}_1 + [1.5, 3]\tilde{x}_2 &\leq [4, 6] \\
 [1, 3]\tilde{x}_1 + [2.5, 3.5]\tilde{x}_2 &\leq 12 \\
 \tilde{x}_1, \tilde{x}_2 &\geq 0
 \end{aligned}$$

*Problem 2.*

$$\begin{aligned}
 \text{Max} Z_2 &= [0.5, 2]\tilde{x}_1 + [-1.5, -1]\tilde{x}_2 \\
 \text{S.T.} \\
 [1, 2]\tilde{x}_1 + [1.5, 3]\tilde{x}_2 &\leq [4, 6] \\
 [1, 3]\tilde{x}_1 + [2.5, 3.5]\tilde{x}_2 &\leq 12 \\
 \tilde{x}_1, \tilde{x}_2 &\geq 0
 \end{aligned}$$

Both problems (1) and (2) are then solved based on the above method. Consider the problem (1). This problem further is transformed into a multi-objective linear programming model as follows:

$$\begin{aligned}
 \text{Max} \underline{Z}_1 &= \underline{x}_1 - \underline{x}_2 \\
 \text{Max} Z_{1c} &= \frac{1}{2}\underline{x}_1 + \frac{3}{2}\bar{x}_1 - \frac{1}{2}\underline{x}_2 + \frac{3}{4}\bar{x}_2 \\
 \text{S.T.} \\
 2\bar{x}_1 + 3\bar{x}_2 &\leq 6 \\
 \frac{1}{2}\underline{x}_1 + \bar{x}_1 + \frac{3}{4}\underline{x}_2 + \frac{3}{2}\bar{x}_2 &\leq 5 \\
 3\bar{x}_1 + 3.5\bar{x}_2 &\leq 12 \\
 \frac{1}{2}\underline{x}_1 + \frac{3}{2}\bar{x}_1 + \frac{5}{4}\underline{x}_2 + \frac{7}{4}\bar{x}_2 &\leq 12 \\
 \underline{x}_2, \bar{x}_1, \underline{x}_2, \bar{x}_2 &\geq 0
 \end{aligned}$$

Solving the  $\underline{Z}_1$  and  $Z_{1c}$  problems distinctly, by the proposed method, will result in  $\tilde{Z}_1^* = [3, 9]$  Similarly, the problem (2) is analyzed and its optimal objective function is obtained as  $\tilde{Z}_2^* = [1.5, 6]$ . Now, the membership functions of objective functions are constituted as follows:

$$\tilde{\mu}_1(x) = \begin{cases} 1 & \text{if } \tilde{z}_1(x) \geq 9 \\ \frac{\tilde{z}_1(x) - 3}{6} & \text{if } \tilde{z}_1(x) \leq 9 \end{cases} \quad \tilde{\mu}_2(x) = \begin{cases} 1 & \text{if } \tilde{z}_2(x) \geq 6 \\ \frac{\tilde{z}_2(x) - 1.5}{4.5} & \text{if } \tilde{z}_2(x) \leq 6 \end{cases}$$

Consequently, the problem is transformed as follows based on model (30):

$$\begin{aligned} & \text{Max} \left[ \frac{5}{18}, \frac{17}{18} \right] \tilde{x}_1 + \left[ -\frac{1}{2}, \frac{1}{36} \right] \tilde{x}_2 - \frac{5}{6} \\ & \text{S.T.} \\ & [1, 3] \tilde{x}_1 + [-1, 1.5] \tilde{x}_2 \geq 3 \\ & [0.5, 2] \tilde{x}_1 + [-1.5, -1] \tilde{x}_2 \geq 1.5 \\ & [1, 2] \tilde{x}_1 + [1.5, 3] \tilde{x}_2 \leq [4, 6] \\ & [1, 3] \tilde{x}_1 + [2.5, 3.5] \tilde{x}_2 \leq 12 \\ & \tilde{x}_1, \tilde{x}_2 \geq 0 \end{aligned} \tag{32}$$

The model (32) is an interval linear programming model which can be solved by first finding its optimum lower bound and center. Then, a bi-objective function is constructed and is solved with a simple weighted summation model with equal weights. The optimal solution is obtained as  $\tilde{x}_1^* = [3, 3]$ ,  $\tilde{x}_2^* = [0, 0]$ ,  $\tilde{Z}_1^* = [3, 9]$  and  $\tilde{Z}_2^* = [1.5, 6]$ .

*Example 2.* Das *et al.* (1999) solved the following multiobjective transportation problem by their proposed method:

$$\begin{aligned} \text{Min} \tilde{Z}_1 &= \sum_{i=1}^3 \sum_{j=1}^4 \tilde{c}_{ij}^1 x_{ij} \\ \text{Min} \tilde{Z}_2 &= \sum_{i=1}^3 \sum_{j=1}^4 \tilde{c}_{ij}^2 x_{ij} \\ & \text{S.T.} \\ & \sum_{j=1}^4 x_{1j} = 8, \quad \sum_{j=1}^4 x_{2j} = 19, \quad \sum_{j=1}^4 x_{3j} = 17, \\ & \sum_{i=1}^3 x_{i1} = 11, \quad \sum_{i=1}^3 x_{i2} = 3, \quad \sum_{i=1}^3 x_{i3} = 14, \quad \sum_{i=1}^3 x_{i4} = 16, \\ & x_{ij} \geq 0, \quad i = 1, 2, 3, \quad j = 1, 2, 3, 4 \end{aligned} \tag{33}$$

where:

$$C^1 = \begin{bmatrix} [1, 2] & [1, 3] & [5, 9] & [4, 8] \\ [1, 2] & [7, 10] & [2, 6] & [3, 5] \\ [7, 9] & [7, 11] & [3, 5] & [5, 7] \end{bmatrix}, \quad C^2 = \begin{bmatrix} [3, 5] & [2, 6] & [2, 4] & [1, 5] \\ [4, 6] & [7, 9] & [7, 10] & [9, 11] \\ [4, 8] & [1, 3] & [3, 6] & [1, 2] \end{bmatrix}$$

Das *et al.* (1999) determined the Pareto optimal solution of the above problem as  $x_{12} = 3, x_{14} = 5, x_{21} = 11, x_{23} = 4.14, x_{24} = 3.86, x_{33} = 9.86$ , and  $x_{34} = 7.14$ . Also,  $\tilde{Z}_1^* = [119.14, 214.42]$  and  $\tilde{Z}_2^* = [180.64, 214.1]$ .

Based on the proposed method, the optimal ranges of objective are determined  $Z_{1c}^* = 148.5, \tilde{Z}_1^* = 187, Z_{2c}^* = 171$  and  $\tilde{Z}_2^* = 211$ . Therefore,  $\tilde{Z}_1^* = [110, 187]$  and  $\tilde{Z}_2^* = [131, 211]$ . Then, the membership functions are constructed as follows:

$$\tilde{\mu}_1(x) = \begin{cases} 1, & \tilde{Z}_1 \leq 110 \\ \frac{187 - \tilde{Z}_1}{77}, & \tilde{Z}_1 \geq 110 \end{cases}, \quad \tilde{\mu}_2(x) = \begin{cases} 1, & \tilde{Z}_2 \leq 133 \\ \frac{211 - \tilde{Z}_2}{78}, & \tilde{Z}_2 \geq 133 \end{cases}$$

Now, developing the model (30) as an interval linear programming model and solving it by the described method, the following solutions are achieved:

- $x_{12} = 3, x_{14} = 5, x_{21} = 11, x_{23} = 8, x_{33} = 6$ , and  $x_{34} = 11$ , with  $\tilde{Z}_1^{*1} = [123, 226]$  and  $\tilde{Z}_2^{*1} = [140, 247]$ ,
- $x_{12} = 3, x_{13} = 5, x_{21} = 11, x_{23} = 8, x_{33} = 1$ , and  $x_{34} = 16$ , with  $\tilde{Z}_1^{*2} = [138, 241]$  and  $\tilde{Z}_2^{*2} = [135, 222]$ ,

Applying the grey possibility degree of Li *et al.* (2007) to compare the obtained results, it can be seen that  $\tilde{Z}_1^{*1} > \tilde{Z}_1^*, \tilde{Z}_2^{*1} < \tilde{Z}_2^*, \tilde{Z}_1^{*2} < \tilde{Z}_1^*$ , and  $\tilde{Z}_2^{*2} > \tilde{Z}_2^*$  which implies the Pareto optimality of the obtained solutions. Also, it is clear that the integrity characteristic of transportation problems solution is hold in the proposed method.

*Example 3.* Oliveira and Antunes (2007) solved the following interval MOLP with Urli and Nadeau (1992) method:

$$\begin{aligned} \text{Min } \tilde{Z}_1 &= [0.5, 1.8]x_1 + [-0.5, 0.5]x_2 \\ \text{Min } \tilde{Z}_2 &= [0.3, 0.8]x_1 + [1, 1.2]x_2 \\ \text{S.T.} \\ [1.5, 2.5]x_1 + [0.5, 1]x_2 &\leq [6, 10] \\ [0.5, 2]x_1 + [3, 6]x_2 &\leq [14, 16] \\ x_1, x_2 &\geq 0 \end{aligned} \tag{34}$$

Solving the problem (34) with Urli and Nadeau (1992), the following solution is obtained:  $x_1 = 3.166, x_2 = 1.188, \tilde{Z}_1^1 = [0.989, 6.2928]$ , and  $\tilde{Z}_2^1 = [2.1378, 3.9584]$ . Solving this problem with the proposed method, the obtained results are as follows:  $x_1 = 3.384615$ ,

$x_2 = 1.538462$ ,  $\tilde{Z}_1^2 = [0.9230765, 6.954317]$ , and  $\tilde{Z}_2^2 = [2.5538465, 4.5538464]$ . For both objectives, it follows that  $\tilde{Z}_1^2 > \tilde{Z}_1^1$  and  $\tilde{Z}_2^2 > \tilde{Z}_2^1$ . In fact the proposed method achieved a preferred solution in both objectives.

## 6. Conclusion

In this paper, the problem of linear multi-objective programming is considered when the model parameters, or coefficients, are stated as interval numbers. The uncertainty is an inevitable characteristic of mathematical modeling of practical systems. Often, the user does not have enough information to determine an exact value for the required values in a model. In this case, the user has to estimate these parameters. Interval numbers provide a framework for expressing the systems' information as interval numbers, rather than crisp and deterministic numbers. This framework provides a great flexibility in modeling the uncertain systems. In this paper, the problem of multi-objective programming is considered, where all of the model's parameters and coefficients are stated as interval numbers. The proposed approach, introduce a process where an interval MOLP problem is transformed into a single objective linear programming model which maximizes the sum of membership functions of different interval objectives. This single objective program itself is an interval linear programming model which can be solved by transforming it into a bi-objective linear model and this model can be solved by current linear multi-objective programming approaches. Also, the efficiency of obtained solution based on the proposed method is proved. Finally, application of the proposed method is examined in two numerical examples. The proposed approach provides a simple, logical and clear framework to deal with interval MOLP problems.

## References

- Barichard, V., Ehrgot, M., Gandibleux, X. and T'Kindt, V. (2009), *Multiobjective Programming and Goal Programming: Theoretical Results and Practical Applications*, Springer, Berlin.
- Chanas, S. and Kuchta, D. (1996), "Multi objective programming in optimization of interval objective functions – a generalized approach", *European Journal of Operational Research*, Vol. 94 No. 3, pp. 594-598.
- Chen, Z., Chen, Q., Chen, W. and Wang, Y. (2004), "Grey linear programming", *Kybernetes*, Vol. 33 No. 2, pp. 238-246.
- Chinneck, J.W. and Ramadan, K. (2000), "Linear programming with interval coefficients", *Journal of the Operational Research Society*, Vol. 51 No. 2, pp. 209-220.
- Climaco, J. (Ed.) (1997), *Multicriteria Analysis*, Springer, New York, NY.
- Cohon, J.L. (2004), *Multi Objective Programming and Planning*, Dover Publication, New York, NY.
- Das, S.K., Goswami, A. and Alam, S.S. (1999), "Multiple objective transportation problem with interval cost, source and destination parameters", *European Journal of Operational Research*, Vol. 117 No. 1, pp. 100-112.
- Huang, G.H. (1994), "Grey mathematical programming and its application to municipal/solid waste management planning", PhD dissertation, Department of Civil Engineering, McMaster University, Hamilton.
- Ida, M. (1999), "Necessary efficient test in interval multi objective linear programming", *Proceedings of the Eighth International Fuzzy Systems Association World Congress, Taipei, Taiwan*, pp. 500-504.

- Ida, M. (2000a), "Efficient solution generation for multiple objective linear programming and uncertain coefficients", *Proceedings of the 8th Bellman Continuum, Taiwan*, pp. 132-136.
- Ida, M. (2000b), "Interval multi objective programming and mobile robot path planning", in Mohammadian, M. (Ed.), *New Frontiers in Computational Intelligence and Its Applications*, IOS Press, Amsterdam, pp. 313-322.
- Ida, M. (2005), "Efficient solution generation for multiple objective linear programming based on extreme ray generation method", *European Journal of Operational Research*, Vol. 160, pp. 242-251.
- Inuiguchi, M. and Kume, Y. (1991), "Goal programming problems with interval coefficients and target intervals", *European Journal of Operational Research*, Vol. 52 No. 3, pp. 345-360.
- Inuiguchi, M. and Sakawa, M. (1995), "Minimax regret solution to linear programming problems with an interval objective function", *European Journal of Operational Research*, Vol. 86 No. 3, pp. 526-536.
- Ishibuchi, H. and Tanaka, H. (1990), "Multi objective programming in optimization of the interval objective function", *European Journal of Operational Research*, Vol. 48 No. 2, pp. 219-225.
- Jimenez, M. and Bilbao-Terol, A. (2009), "Pareto-optimal solutions in fuzzy multi-objective linear programming", *Fuzzy Sets and Systems*, Vol. 160 No. 18, pp. 2714-2721.
- Li, G.D., Yamaguchi, D. and Nagai, M. (2007), "A grey-based decision-making approach to the supplier selection problem", *Mathematical and Computer Modelling*, Vol. 46 Nos 3/4, pp. 573-581.
- Liu, S. and Lin, Y. (2006), *Grey Information: Theory and Practical Applications*, Springer, London.
- Liu, S., Sheng, K. and Forrest, J. (2012), "On uncertain systems and uncertain models", *Kybernetes*, Vol. 41 Nos 5/6, pp. 548-558.
- Moore, R.E. (1966), *Interval Analysis*, Prentice-Hall, Englewood Cliffs, NJ.
- Moore, R.E., Kearfott, R.B. and Cloud, M.J. (2009), *Introduction to Interval Analysis*, SIAM Press, Philadelphia, PA.
- Nocedal, J. and Wright, S.J. (1999), *Numerical Optimization*, Springer, New York, NY.
- Oliveira, C. and Antunes, C.H. (2007), "Multiple objective linear programming models with interval coefficients – an illustrated overview", *European Journal of Operational Research*, Vol. 181 No. 3, pp. 1434-1463.
- Razavi Hajiagha, S.H., Akrami, H. and Hashemi, S.S. (2012), "A multi objective programming approach to solve grey linear programming", *Grey Systems: Theory and Application*, Vol. 2 No. 2, EarlyCite Article.
- Sengupta, A., Pal, T.K. and Chakraborty, D. (2001), "Interpretation of inequality constraints involving interval coefficients and a solution to interval linear programming", *Fuzzy Sets and Systems*, Vol. 119 No. 1, pp. 129-138.
- Steuer, R.E. (1980), "Algorithms for linear programming problems with interval objective function coefficients", *Mathematics of Operations Research*, Vol. 6 No. 3, pp. 333-348.
- Steuer, R.E. (1986), *Multiple Criteria Optimization: Theory, Computation and Application*, Wiley, New York, NY.
- Tamiz, M. (1996), *Multi Objective Programming and Goal Programming: Theories and Applications*, Springer, Berlin.
- Urli, B. and Nadeau, R. (1992), "An interactive method to multiobjective linear programming problems with interval coefficients", *INFOR*, Vol. 30, pp. 127-137.

- Wang, M.L. and Wang, H.F. (2001a), "Decision analysis of the interval-valued multi objective linear programming problems", in Koksalan, M. and Zionts, S. (Eds), *Multiple Criteria Decision Making in the New Millennium*, Notes in Economics and Mathematical Systems, Vol. 507, Springer, Berlin, pp. 210-218.
- Wang, M.L. and Wang, H.F. (2001b), "Interval analysis of a fuzzy multi objective linear programming", *International Journal of Fuzzy Systems*, Vol. 34, pp. 558-568.
- Werners, B. (1987a), "An interactive fuzzy programming system", *Fuzzy Sets and Systems*, Vol. 23, pp. 131-147.
- Werners, B. (1987b), "Interactive multiple objective programming subject to flexible constraints", *European Journal of Operational Research*, Vol. 31, pp. 342-349.
- Yovits, M.C. (1984), *Advances in Computers*, Vol. 23, Academic Press, Orlando, FL.

#### About the authors

Seyed Hossein Razavi Hajiagha is Assistant Professor at the Department of Systemic and Productivity Studies, Institute for Trade Studies and Research, Tehran, Iran. He received a PhD in Production and Operation Management in 2012. He is author and co-author of about ten scientific papers and his research interests include multiple criteria analysis, decision-making theories, data envelopment analysis and industrial problems mathematical modelling. Seyed Hossein Razavi Hajiagha is the corresponding author and can be contacted at: s.hossein.r@gmail.com

Hannan Amoozad Mahdiraji has a PhD in Production and Operation Management and is an Assistant Professor at the Kashan branch Islamic Azad University in Iran, He is Interested in supply chain modeling, especially game theory applications in coordination and cooperation, multiple criteria decision-making techniques and also fuzzy and grey systems. He has published several scientific papers and also contributed at many international conferences.

Shide Sadat Hashemi has an MA in Production and Operation Management and has worked as a researcher since 2009. Her research interests include operation management, multiple criteria decision making and data envelopment analysis. She is author and co-author of six scientific papers.